Ex.10 Show that, if $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors, and $\vec{p}\cdot\vec{a} = \vec{p}\cdot\vec{b} = \vec{p}\cdot\vec{c} = 0$, then $\vec{p}$ is the zero vector.

Sol. Since $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ it is normal to the plane of $\vec{a}$ and $\vec{b}$. Then, since $\vec{p}\cdot\vec{c} = 0$, $\vec{c}$ must lie in the plane of $\vec{a}$ and $\vec{b}$. But this is contrary to the data. Hence $\vec{p}$ must be zero.

Ex.11 Show that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

Sol. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of A, B, C and $\vec{h}$ that of the intersection, H, of the perpendiculars from B and C. then $(\vec{h} - \vec{b})\cdot(\vec{a} - \vec{c}) = 0$, $(\vec{h} - \vec{c})\cdot(\vec{b} - \vec{a}) = 0$.

Addition of these shows that $(\vec{h} - \vec{a})\cdot(\vec{a} - \vec{c}) = 0$, so that AH is perpendicular to BC, and the theorem is proved.

Ex.12 Show that the perpendicular bisectors of the sides of a triangle are concurrent.

Sol. Let K be the intersection of the perpendicular bisectors of AB and AC. Then, with the notation of previous Ex, $\left(\vec{k} - \frac{\vec{a}+\vec{b}}{2}\right)\cdot(\vec{a} - \vec{b}) = 0$, $\left(\vec{k} - \frac{\vec{a}+\vec{c}}{2}\right)\cdot(\vec{c} - \vec{a}) = 0$. Addition of these shows that $\left(\vec{k} - \frac{\vec{c}+\vec{b}}{2}\right)\cdot(\vec{c} - \vec{b}) = 0$, so that K is also on the perpendicular bisector of BC.

Ex.13 In a tetrahedron, if two pairs of opposite edges are perpendicular, the third pair are also perpendicular to each other; and the sum of the squares of two opposite edges is the same for each pair.

Sol. We have $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$ and $\overrightarrow{CB} = \vec{b} - \vec{c}$. Hence if BD is perpendicular to CA, $\vec{b}\cdot(\vec{c} - \vec{a}) = 0$, i.e. $\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{b}$ And similarly, if DA is perpendicular to BC, $\vec{a}\cdot(\vec{b} - \vec{c}) = 0$ i.e. $\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{a}$

Thus $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a}$, whence $\vec{c}\cdot(\vec{a} - \vec{b}) = 0$,
showing that DC is perpendiuclar to BA.
Further, the sum of the squares on BD and CA is $b^2 + (\vec{c} - \vec{a})^2 = b^2 + c^2 + a^2 - 2\vec{a}\cdot\vec{c}$,
and, in virtue of (i), this is the same as in the other two cases.

Remark :

(i) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta\ (0 \le \theta \le \pi)$. note that if θ is acute then $\vec{a}\cdot\vec{b} > 0$ & if θ is obtuse then $\vec{a}\cdot\vec{b} < 0$

(ii) $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

(iii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

(iv) $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad(\vec{a} \ne 0 \quad \vec{b} \ne 0)$

(v) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(vi) Projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

(vii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ $\quad 0 \le \phi \le \pi$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\ ,\quad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

(ix) Maximum value of $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$ **(x)** Minimum values of $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

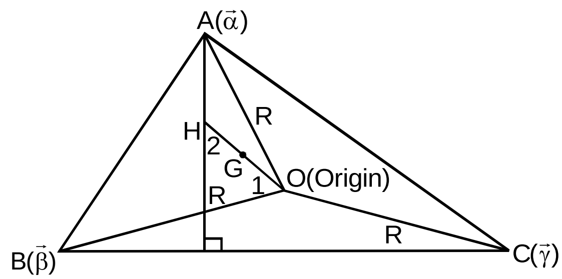
(xi) Any vector $\vec{a}$ can be written as , $\vec{a} = (\vec{a}\cdot\hat{i})\,\hat{i} + (\vec{a}\cdot\hat{j})\,\hat{j} + (\vec{a}\cdot\hat{k})\,\hat{k}$.

(xii) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$.

Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda\,(\hat{a} + \hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda\,(\hat{a} - \hat{b})$, $\lambda \in R^+$.

Ex.14 In a Δ ABC if 'O' is the circumcentre, H is the orthocentre and R is the radius of the circle circumscribing the triangle ABC, then prove that ;

(i) $\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}$ **(ii)** $\left|\vec{OH}\right|^2 = 9\,R^2 - (a^2 + b^2 + c^2)$ **(iii)** $\left|\vec{AH}\right| = 2\,R\,|\cos A|$



Sol. **(i)** position vector of G are $\dfrac{\vec{\alpha} + \vec{\beta} + \vec{\gamma}}{3} = \vec{OG}$

$\Rightarrow \vec{OH} = 3\vec{OG} = \vec{\alpha} + \vec{\beta} + \vec{\gamma}$

$= \vec{OA} + \vec{OB} + \vec{OC}$

(ii) $\left|\vec{OH}\right|^2 = (\vec{\alpha} + \vec{\beta} + \vec{\gamma})^2 = 3R^2 + 2R^2[\Sigma \cos 2A]$ $= 3\,R^2 + 2\,R^2\,[3 - 2\,(\Sigma \sin^2 A]$

$= 9\,R^2 - 4\,R^2\left(\dfrac{a^2}{4R^2} + \dfrac{b^2}{4R^2} + \dfrac{c^2}{4R^2}\right)$ $= 9\,R^2 - (a^2 + b^2 + c^2)$

(iii) $\vec{AH}$ = position vector of $\vec{H}$ – position vector of $\vec{A}$ $= \vec{\alpha} + \vec{\beta} + \vec{\gamma} - \vec{\alpha} = \vec{\beta} + \vec{\gamma}$

$\left|\vec{AH}\right|^2 = (\vec{\beta} + \vec{\gamma})^2 = 2\,R^2 + 2\,R^2 \cos 2A = 4\,R^2 \cos^2 A$

Ex.15 Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60º. Suppose that $|\vec{u} - \hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u} - 2\hat{i}|$ where $\hat{i}$ is the unit vector angle x-axis then $|\vec{u}|$ has the value equal to $\sqrt{a} - \sqrt{b}$ where a, b $\in$ N, find the value $(a + b)^3 + (a - b)^3$.

Sol. Let $\vec{u} = x\hat{i} + \sqrt{3}\,x\,\hat{j}$; $|\vec{u}| = 2x,\ x > 0$

now $|\vec{u}|\,\left|\vec{u} - 2\hat{i}\right| = \left|\vec{u} - \hat{i}\right|^2 \Rightarrow 2\,|x|\,\sqrt{(x-2)^2 + 3x^2} = [(x - 1)^2 + 3x^2]$

$\Rightarrow 2\,|x|\,\sqrt{4x^2 - 4x + 4} = 4x^2 - 2x + 1 \qquad \Rightarrow \qquad 4\,|x|\,\sqrt{x^2 - x + 1} = 4x^2 - 2x + 1$

square $\Rightarrow \quad 16x^2(x^2 - x + 1) = 16x^4 + 4x^2 + 1 - 16x^3 - 4x + 8x^2 \qquad \Rightarrow \qquad 16x^2 = 12x^2 + 1 - 4x$

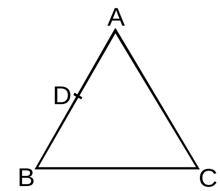
$\Rightarrow \quad 4x^2 + 4x - 1 = 0 \Rightarrow x = \dfrac{-4 \pm \sqrt{16+16}}{8} = \dfrac{-4 \pm 4\sqrt{2}}{8} = \dfrac{-1 \pm \sqrt{2}}{2}$ or $\dfrac{-(1+\sqrt{2})}{2}$

$\Rightarrow \quad 2x = \sqrt{2} - 1$ or $-(\sqrt{2} + 1) \rightarrow$ rejected

hence $|\vec{u}| = \sqrt{2} - 1 = \sqrt{2} - \sqrt{1} \quad \Rightarrow \quad a = 2;\ b = 1 \Rightarrow (a+b)^3 + (a-b)^3 = 27 + 1 = 28$

Ex.16 In a triangle ABC, AB = AC and O is its circumcentre. Also D is the midpoint of AB and E is the centroid of triangle ACD. Using vector method, prove that CD is perpendicular to OE.

Sol. Take A as the initial point and the p.v. of B, C and O as $\vec{b}, \vec{c}$ and $\vec{r}$ respectively.



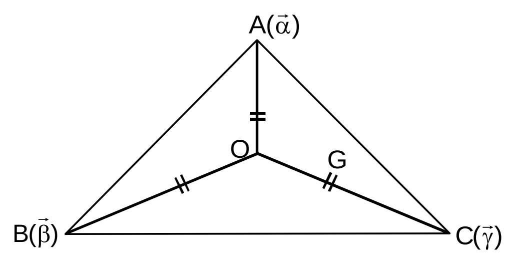
The p.v. of D is $\dfrac{\vec{b}}{2}$ and that of E $\dfrac{2\vec{c} + \vec{r}}{6}$. Since O is the circumcentre of

ΔABC, $|\vec{r}| = |\vec{r} - \vec{c}| = |\vec{r} - \vec{b}| \quad \Rightarrow \vec{r}.\vec{b} = \vec{r}.\vec{c} = \dfrac{|\vec{b}|^2}{2} = \dfrac{|\vec{c}|^2}{2}$.

Hence $\overrightarrow{CD}.\overrightarrow{OE} = \left(\vec{c} - \dfrac{\vec{b}}{2}\right)\left(\dfrac{2\vec{c} + \vec{b}}{6} - \vec{r}\right) = \dfrac{4|\vec{c}|^2 - |\vec{b}|^2}{12} - \vec{r}.\vec{c} + \dfrac{1}{2}\vec{r}.\vec{b} = \dfrac{|\vec{b}|^2}{4} - \dfrac{|\vec{b}|^2}{2} + \dfrac{|\vec{b}|^2}{4} = 0 \quad \Rightarrow \quad CD \perp OE.$

Ex.17 Prove using vectors that the distance of the circumcenter of the Δ ABC from the centroid is $\sqrt{R^2 - \dfrac{1}{9}(a^2 + b^2 + c^2)}$ where R is the circumradius.

Sol. circumcenter is 'O' $\left|\overrightarrow{OG}\right|^2 = \dfrac{1}{9}\left|\vec{\alpha} + \vec{\beta} + \vec{\gamma}\right|^2$



$= \dfrac{1}{9}\left(\left|\vec{\alpha}\right|^2 + \left|\vec{\beta}\right|^2 + \left|\vec{\gamma}\right|^2 + 2\vec{\alpha}.\vec{\beta} + 2\vec{\beta}.\vec{\gamma} + 2\vec{\gamma}.\vec{\alpha}\right)$

$= \dfrac{1}{9}[3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C]$

$= \dfrac{1}{9}[3R^2 + 2R^2(3 - 2\sin^2 A)] = \dfrac{1}{9}\left[9R^2 - 4R^2\left(\dfrac{a^2}{4R^2} + \dfrac{b^2}{4R^2} + \dfrac{c^2}{4R^2}\right)\right]$

Ex.18 Let ABC be a triangle with AB = AC. If D is the mid point of BC, E the foot of the perpendicular drawn from D to AC and F the mid point of DE, use vector methods to prove that AF is perpendicular to BE.

Sol. Let D be the initial point. Let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$, $-\vec{b}$ respectively.

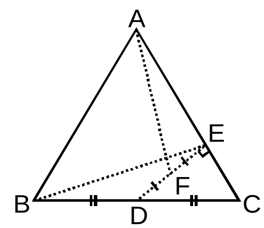
It is given that AB = AC $\Rightarrow$ AD is perpendicular to BC $\Rightarrow \vec{a}.\vec{b} = 0$,

If E divides AC in the ratio 1 : λ, then p.v. of E is $\dfrac{\lambda\vec{a} - \vec{b}}{1+\lambda}$

Since DE is perpendicular to AC, $\overrightarrow{DE}.\overrightarrow{AC} = 0 \quad \Rightarrow \quad \left(\dfrac{\lambda\vec{a} - \vec{b}}{1+\lambda}\right).(\vec{a} + \vec{b}) = 0$

$\Rightarrow$ l $|\vec a|^2 = |\vec b|^2 = 0 \Rightarrow \lambda = \dfrac{|\vec b|^2}{|\vec a|^2}$, so that p.v. of E is $\dfrac{|\vec b|^2\vec a - |\vec a|^2\vec b}{|\vec a|^2 + |\vec b|^2} \Rightarrow$ p.v. of F is $\dfrac{|\vec b|^2\vec a - |\vec a|^2\vec b}{2\left(|\vec a|^2 + |\vec b|^2\right)}$

Vector $\overrightarrow{AF} = \left(\dfrac{|\vec b|^2\vec a - |\vec a|^2\vec b}{2\left(|\vec a|^2 + |\vec b|^2\right)}\right) - \vec a = \dfrac{-(2|\vec a|^2 + |\vec b|^2)\vec a - |\vec a|^2\,\vec b}{2(|\vec a|^2 + |\vec b|^2)}$



Vector $\overrightarrow{BE} = \left(\dfrac{|\vec b|^2\vec a - |\vec a|^2\vec b}{|\vec a|^2 + |\vec b|^2}\right) - \vec b = \dfrac{-(2|\vec a|^2 + |\vec b|^2)\vec b + |\vec b|^2\,\vec a}{|\vec a|^2 + |\vec b|^2}$

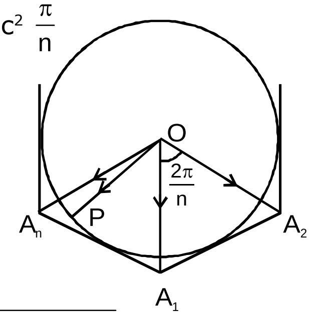
Hence $\overrightarrow{AF}.\overrightarrow{BE} = \dfrac{(|\vec a|^2\,|\vec b|^2\,(-2|\vec a|^2 - |\vec b|^2 + 2|\vec a|^2 + |\vec b|^2)}{2(|\vec a|^2 + |\vec b|^2)^2} = 0 \quad \Rightarrow$ AF is perpendicular to BE.

Ex.19 $A_1, A_2, \ldots\ldots\ldots, A_n$ be an n sided regular polygon circumscribed over a circle of radius r. If P be any point on circle. Using vectors prove that $PA_1 + PA_2 + \ldots\ldots\ldots + PA_n \le \sqrt{3nr^2\left(1 + \sec^2\dfrac{\pi}{n}\right)}$

Sol. Let centre be point of reference O (origin) $\Rightarrow \quad \overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \ldots\ldots\ldots + \overrightarrow{OA_n} = \vec 0$ ....(1)

$\sum\left(\overrightarrow{PA_i}\right)^2 = \sum\left(\overrightarrow{OP} - \overrightarrow{OA_i}\right)^2 = \sum\left(\overrightarrow{OP}\right)^2 + \sum\left(\overrightarrow{OA_i}\right)^2 - 2\sum \overrightarrow{OP}.\overrightarrow{OA_i} = nr^2 + nr^2\sec^2\dfrac{\pi}{n}$

and $|\overrightarrow{OA_i}| = r\sec\dfrac{\pi}{n}$



Now, $\left|\sum(\overrightarrow{PA_i})\right|^2 = \sum|(\overrightarrow{PA_i})|^2 + 2\sum|\overrightarrow{PA_1}|.|\overrightarrow{PA_2}|$

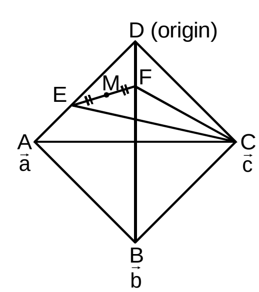
Also, $\sum(\overrightarrow{PA_i})^2 \ge \sum|\overrightarrow{PA_1}|.|\overrightarrow{PA_2}| \quad \Rightarrow \quad 3\sum(\overrightarrow{PA_i})^2 \ge \left(\sum|\overrightarrow{PA_i}|\right)^2$

$3nr^2\left(1 + \sec^2\dfrac{\pi}{n}\right) \ge \left(\sum|\overrightarrow{PA_i}|\right)^2 \quad \Rightarrow \quad PA_1 + PA_2 + \ldots\ldots\ldots + PA_n \le \sqrt{3nr^2\left(1 + \sec^2\dfrac{\pi}{n}\right)}$

Ex.20 The length of the edge of the regular tetrahedron D – ABC is 'a'. Point E and F are taken on the edges AD and BD respectively such that E divides $\overrightarrow{DA}$ and F divides $\overrightarrow{BD}$ in the ratio 2 : 1 each. Then find the area of triangle CEF.

Sol. We have $|\vec a| = |\vec b| = |\vec c| = |\vec b - \vec a| = |\vec b - \vec c| = |\vec c - \vec a| = a$ and $\hat{\vec a\,\vec b} = \hat{\vec b\,\vec c} = \hat{\vec c\,\vec a} = \overrightarrow{AB}\,\hat{}\,\overrightarrow{AC} = \overrightarrow{CA}\,\hat{}\,\overrightarrow{CB} = \overrightarrow{BA}\,\hat{}\,\overrightarrow{BC} = \dfrac{\pi}{3}$

$(\overrightarrow{EF})^2 = \dfrac{a^2}{3} \quad \Rightarrow \quad |\overrightarrow{EF}| = \dfrac{a}{\sqrt3}$



$|\overrightarrow{CF}| = |\overrightarrow{CE}| = \dfrac{\sqrt7\,a}{3}$, $|\overrightarrow{CM}| = \dfrac{5a}{6}$ where M is the middle point of EF.

Area (DCEF) $= \dfrac12\,|\overrightarrow{EF}|\,|\overrightarrow{CM}| = \dfrac12 \cdot \dfrac{a}{\sqrt3} \cdot \dfrac{5a}{6} = \dfrac{5a^2}{12\sqrt3}$

Ex.21 Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $\left(\overrightarrow{PA}.\overrightarrow{PB}\right) + 3\left(\overrightarrow{OA}.\overrightarrow{OB}\right) = 0$. If the maximum and minimum values of $\left|\overrightarrow{PA}\right|\left|\overrightarrow{PB}\right|$ are M and m respectively then find the value of $M^2 + m^2$.

Sol. Let P be (x, y). $\overrightarrow{PA} = (1 - x)\hat{i} - y\hat{j}$; $\overrightarrow{PB} = (-1 - x)\hat{i} - y\hat{j}$

$\therefore \quad \left(\overrightarrow{PA}.\overrightarrow{PB}\right) = ((x-1)\hat{i} + y\hat{j}).((x+1)\hat{i} + y\hat{j}) = (x^2 - 1) + y^2$ also $\quad 3\left(\overrightarrow{OA}.\overrightarrow{OB}\right) = 3\hat{i}.(-\hat{i}) = -3$

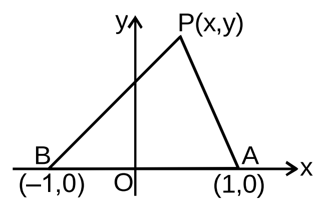
hence $\left(\overrightarrow{PA}.\overrightarrow{PB}\right) + 3\left(\overrightarrow{OA}.\overrightarrow{OB}\right) = 0 \Rightarrow x^2 - 1 + y^2 - 3 = 0 = 0 \Rightarrow x^2 + y^2 = 4$ ....(1)

which gives the locus of P i.e. P move on a circle with centre (0, 0) and radius 2.

now $\left|\overrightarrow{PA}\right|^2 = (x - 1)^2 + y^2$; $\left|\overrightarrow{PB}\right|^2 = (x + 1)^2 + y^2$

$\therefore \quad \left|\overrightarrow{PA}\right|^2\left|\overrightarrow{PB}\right|^2 = (x^2 + y^2 - 2x + 1)(x^2 + y^2 + 2x + 1) = (5 - 2x)(5 + 2x)$ [using $x^2 + y^2 = 4$]

$\therefore \quad \left|\overrightarrow{PA}\right|^2\left|\overrightarrow{PB}\right|^2 = 25 - 4x^2$ subject to $x^2 + y^2 = 4$



$\left.\left|\overrightarrow{PA}\right|^2\left|\overrightarrow{PB}\right|^2\right|_{min.} = 25 - 16 = 9$; (when x = 2 or –2)

and $\quad \left.|\overrightarrow{PA}|^2|\overrightarrow{PB}|^2\right|_{max.} = 25 - 0 = 25$ (when x = 0)

$3 \le |\overrightarrow{PA}||\overrightarrow{PB}| \le 5$. hence M = 5 and m = 3 $\Rightarrow M^2 + m^2 = 34$ Ans.

F. LINEAR COMBINATIONS

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots\ldots$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots\ldots$ for any x, y, z .... $\in$ R. We have the following results :

(a) If $\vec{a}, \vec{b}$ are non zero., non-collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'$; $y = y'$

(b) **Fundamental Theorem :** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. There exist some unique x, y $\in$ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(c) If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors then: $x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x'$, $y = y'$, $z = z'$

(d) **Fundamental Theorem In Space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x, y, z $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1, \vec{x}_2, \ldots\ldots\ldots\ldots \vec{x}_n$ are n non zero vectors, & k_1, k_2 ,........... k_n are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots\ldots k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, k_2 = 0 \ldots\ldots\ldots\ldots k_n = 0$ then we say that vectors $\vec{x}_1\vec{x}_2\ldots\ldots\ldots\vec{x}_n$ are Linearly Independent Vectors.

(f) If $\vec{x}_1\vec{x}_2\ldots\ldots\ldots\vec{x}_n$ are not Linearly Independent then they are said to be Linearly Dependent vectors. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots\ldots k_n\vec{x}_n = 0$ & if there exists at least one $k_r \ne 0$ then $\vec{x}_1\vec{x}_2\ldots\ldots\ldots\vec{x}_n$ are said to be Linearly Dependent.

Note 1 : If $k_r \neq 0$; $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 + \dots\dots\dots + k_r\vec{x}_r + \dots\dots\dots + k_n\vec{x}_n = 0$

$- k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + \dots\dots\dots + k_{r-1}.\vec{x}_{r-1} + k_{r+1}.\vec{x}_{r+1} + \dots\dots\dots + k_n\vec{x}_n$

$- k_r \frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r}\vec{x}_2 + \dots\dots\dots + k_{r-1}.\frac{1}{k_r}\vec{x}_{r-1} + \dots\dots\dots + k_n\frac{1}{k_r}\vec{x}_n$

$\vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots\dots\dots\dots + c_{r-1}\vec{x}_{r-1} + c_1\vec{x}_{r-1} + \dots\dots\dots\dots\dots + c_n\vec{x}_n$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors.

$\vec{x}_1$, $\vec{x}_2$...........$\vec{x}_{r-1}$, $\vec{x}_{r+1}$.............$\vec{x}_n$

Hence $\vec{x}_r$ with $\vec{x}_1, \vec{x}_2$...........$\vec{x}_{r-1}$, $\vec{x}_{r+1}$.............$\vec{x}_n$ forms a linearly dependent set of vectors.

Note 2 :

(i) If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a Linear Combination of vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Also, $\vec{a}$, $\hat{i}$, $\hat{j}$, $\hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

(ii) $\hat{i}$, $\hat{j}$, $\hat{k}$ are Linearly Independent set of vectors. For $K_1 i + K_2 j + K_3 k = 0 \Rightarrow K_1 = 0 = K_2 = K_3$.

(iii) Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0$ $\Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq 0$, then the vectors are linearly independent.

(iv) If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}, \vec{b}, \vec{c}] = 0$, conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent.

Ex.22 Show that the points whose position vectors are $\vec{a} + 2\vec{b} + 5\vec{c}, 3\vec{a} + 2\vec{b} + \vec{c}, 2\vec{a} + 2\vec{b} + 3\vec{c}$ are collinear.

Sol. Let the given points be A, B, C and O be the point of reference.

Then $\overrightarrow{OA} = \vec{a} + 2\vec{b} + 5\vec{c}, \overrightarrow{OB} = 3\vec{a} + 2\vec{b} + \vec{c}$ and $\overrightarrow{OC} = 2\vec{a} + 2\vec{b} + 3\vec{c}$

Let us assume that l, m, n be three scalar quantities, such that $\ell\overrightarrow{OA} + m\overrightarrow{OB} + n\overrightarrow{OC}$ ....(i)

where $\ell + m + n = 0$ ....(ii)

Now $\ell(\vec{a} + 2\vec{b} + 5\vec{c}) + m(3\vec{a} + 2\vec{b} + \vec{c}) + n(2\vec{a} + 2\vec{b} + 3\vec{c}) = 0 = 0\vec{a} + 0\vec{b} + 0\vec{c}$.

Comparing the coefficients of $\vec{a}, \vec{b}, \vec{c}$ on both sides, we get

$\ell + 3m + 2n = 0$, $2\ell + 2m + 2n = 0$, $5\ell + m + 3n = 0$

or, $\ell + m + n + 2m + n = 0$ $\Rightarrow$ $2(\ell + m + n) = 0$, $(\ell + m + n) + 4\ell + 2n = 0$

or, $2m + n = 0$, $4\ell + 2n = 0$ from (ii) or, $\ell = \frac{-1}{2}n$, $m = \frac{-1}{2}$ n which satisfy (ii)

Hence the condition of collinearly (i) and (ii) are satisfied. Hence the given points are collinear.

Ex.23 Examine if $\vec{i} - 3\vec{j} + 2\vec{k}, 2\vec{i} - 4\vec{j} - \vec{k}$ and $3\vec{i} + 2\vec{j} - \vec{k}$ are linearly independent or dependent.

Sol. If the vectors are linearly dependent, $\ell\left(\vec{i} - 3\vec{j} + 2\vec{k}\right) + m\left(2\vec{i} - 4\vec{j} - \vec{k}\right) + n\left(3\vec{i} + 2\vec{j} - \vec{k}\right) = \vec{0}$

Where ℓ, m, n are scalars not all zero.

$\Rightarrow$ $\ell + 2m + 3n = 0$ ....(i), $-3\ell - 4m + 2n = 0$ .....(ii), $2\ell - m - n = 0$ ....(iii)

from (i) and (ii) $\Rightarrow$ $\frac{\ell}{16} = \frac{m}{11} = \frac{n}{2} = k$ say $\Rightarrow$ $\ell = 16k$, $m = -11k$, $n = 2k$

These ℓ, m, n do not satisfy (iii) and hence the given system is linearly independent.

G. VECTOR PRODUCT OF TWO VECTORS

Vector quantities are of frequent occurrence, which depend each upon two other vector quantities in such a way as to be jointly proportional to their magnitudes and to the sine of their mutual inclination, and to have a direction perpendicular to each of them. We are therefore led to adopt the following

Definition : The vector product of two vectors $\vec{a}$ and $\vec{b}$, whose directions are inclined at an angle θ, is the vector whose modulus is ab sin θ, and whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$, being positive relative to a rotation from $\vec{a}$ to $\vec{b}$. we write it $\vec{a}\times\vec{b}$ = ab sin $\theta\hat{n}$

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$, $\vec{b}$ having the same direction as the translation of a right-handed screw due to a rotation from $\vec{a}$ to $\vec{b}$. From this it follows that $\vec{b} \times \vec{a}$ has the opposite direction to $\vec{a} \times \vec{b}$, but the same length, so that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

The order of the factors in vector product is not commutative; for a reversal of the order alters the sign of the product.

Consider the parallelogram OAPB whose sides OA, OB have the lengths and directions of $\vec{a}$ and $\vec{b}$ respectively. The area of the figure is ab sin θ, and the vector area OAPB, whose boundary is described in this sense, is represented by ab sin $\theta\,\hat{n} = \vec{a} \times \vec{b}$. This simple geometrical relation will be formed useful. The vector area OBPA is of course represented by b × a.

For two parallel vectors sin θ is zero and their vector product vanishes. The relation a × b = 0 is thus the condition of parallelism of two proper vectors. In particular, r × r = 0 is true for all vectors. If, however, $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a} \times \vec{b}$ is a vector whose modulus is ab, and whose direction is such that $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ form a right-handed system of mutually perpendicular vectors. If $\vec{a}$, $\vec{b}$ are both unit vectors the modulus of $\vec{a} \times \vec{b}$ is the sine of their angle of inclination. For the particular unit vectors, $\hat{i}$, $\hat{j}$, $\hat{k}$

we have $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$,

while $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$, $\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j}$, $\hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}$.

These relations will be constantly employed.

If either factor is multiplied by a number, their product is multiplied by that number. For

$$(m\vec{a}) \times \vec{b} = \text{mab} \sin\theta\,\hat{n} = \vec{a} \times (m\vec{b}).$$

Distributive Law : We shall now show that the distributive law holds for vector products also ;

i.e. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. ....(1)

From the distributive law may be deduced a very useful formula for the vector product $\vec{a} \times \vec{b}$ in terms of rectangular components of the vectors. For, with the usual notation,

$\vec{a} \times \vec{b}\ (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$ ...(2)

in virtue of the relation proved in the preceding Art. We may write this in the determinantal form

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix} \qquad ....(3)$$

This vector has modulus ab sin θ. Hence, on squaring both members of the above equation and dividing by a^2b^2, we find for the sine of the angle between two vectors $\vec{a}$ and $\vec{b}$.

$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}.$$

If l, m, n and l', m', n' are the direction cosines of $\vec{a}$ and $\vec{b}$ respectively, this is equivalent to

$$\sin^2\theta = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

It is worth noticing that if $\vec{b} = \vec{c} + n\vec{a}$, where n is any real number, then

$$\vec{a} \times \vec{b} = \vec{a} \times (\vec{c} + n\vec{a}) = \vec{a} \times \vec{c}.$$

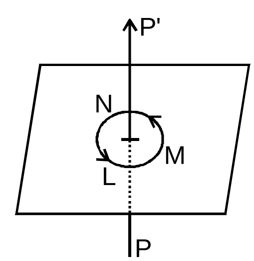
Conversely, if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ it does not follow that $\vec{b} = \vec{c}$, but that $\vec{b}$ differs from $\vec{c}$ by some vector parallel to $\vec{a}$, which may or may not be zero.

Remark :

(i) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\hat{n}$ forms a right handed screw system.

Vector area : Consider the type of vector quantity whose magnitude is an area. Such a quantity is associated with each plane figure, the magnitude being the area of the figure, and the direction that of the normal to the plane of the figure. This vector area therefore specifies both the area and the orientation of the plane figure. But as the direction might be either of two opposite directions along the normal, some convection is necessary. The area clearly has no sign in itself, and can be regarded as positive or negative only with reference to the direction in which the boundary of the figure is described, or the side of the plane from which it is viewed.

Consider the area of the figure bounded by the closed curve LMN, which is regarded being traced out in the direction of the arrows, the normal vector $\overrightarrow{PP'}$, bears to this direction of rotation the same relation as the translation to the direction of rotation of a right-handed screw. The area LMN is regarded as positive relative to the direction of $\overrightarrow{PP'}$.



With this convention a vector area may be represented by a vector normal to the plane of the figure, in the direction relative to which it is positive, and with modulus equal to the measure of the area. The sum of two vector areas represented by $\vec{a}$ and $\vec{b}$ is defined to be the vector area represented by $\vec{a} + \vec{b}$.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

(iii) $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) ($\vec{a} \neq 0$, $\vec{b} \neq 0$) i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(iv) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

(v) $(m\vec{a}) \times \vec{b} = \vec{a} x (m\vec{b}) = m(\vec{a} \times \vec{b})$ where m is a scalar.

(vi) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

(vii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$ **(viii)** $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

(ix) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(xi) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(xii) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(xiii) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle

$\Delta ABC = \dfrac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$. The points A, B & C are collinear if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = 0$

or $\Delta ABC = \dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

(xiv) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$

(xv) Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 = \begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

TEST OF COLLINEARITY : Three points A, B, C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a}+y\vec{b}+a\vec{c}=0$, where x + y + z = 0.

Theorem of Menelaus. If a transversal cuts the side BC, CA, AB of a triangle in the point P, Q, R respectively, the product of the ratios in which P, Q, R divide those sides is equal to −1. Let Q divide CA in the ratio m : n. Then we can find a number ℓ such that R divides AB in the ratio ℓ : m. Consequently

$$(m+n)\,\vec{q} = n\vec{c} + m\vec{a}, \qquad (\ell+m)\vec{r} = m\vec{a} + \ell\vec{b}.$$

Eliminate $\vec{a}$ and write the result in the form $\dfrac{n\vec{c}-\ell\vec{b}}{n-\ell} = \dfrac{(m+n)\vec{q}-(\ell+m)\vec{r}}{n-\ell} = \vec{p}$.

Thus BC and RQ intersect at P, which divides BC in the ratio −n : 1. consequently the product of the three ratios is −1.

Observing that $(n-\ell)\,\vec{p} - (m+n)\,\vec{q} + (\ell+m)\,\vec{r} = 0$ show conversely that, if the product of the above ratios is −1, then P, Q, R are collinear.

Theorem of Desargues. If the lines joining corresponding vertices of two triangles are concurrent, the points of intersection of corresponding sides are collinear.

Let A, B, C correspond to D, E, F respectively. Then BC and EF are corresponding sides. Given AD, BE, CF intersect at a point H, we have relations of the form $\ell\vec{a} + \ell'\vec{b} = m\vec{b} + m'\vec{e} = n\vec{c} + n'\vec{f} = \vec{h}$

where $\ell + \ell' = m + m' = n + n' = 1$ Hence $\dfrac{m\vec{b}-n\vec{c}}{m-n} = \dfrac{m'\vec{e}-n'\vec{f}}{m'-n'} = \vec{p}$

giving the position vector of the point P of intersection of BC and EF. Write down similar expressions for $\vec{q}$ and $\vec{r}$, the intersections of the other pairs of corresponding sides, and verify that

$$(m-n)\vec{p} + (n-\ell)\vec{q} + (\ell-m)\vec{r} = 0,$$

in which the sum of the coefficient is zero. Consequently P, Q, R are collinear.

Theorem of Pascal. If A, B, C are points on one of two intersecting straight lines, and A', B', C' are on the other, then the point P, in which BC' cuts B'C, is colinear with the points Q, R in which CA' cuts C'A, and AB' cuts A'B.

Use the notation and result of Ex.1 and write down the corresponding expressions for $\vec{p}$ and $\vec{q}$, the position vectors of P and Q. Hence, taking H as origin, show that

$$aa'(cc' - bb')\vec{p} + bb'(aa' - cc')\vec{q} + cc'(bb' - aa')\vec{r} = 0,$$

a linear relation in which the sum of the coefficients of $\vec{p}$, $\vec{q}$, $\vec{r}$ is zero. Consequently the points P, Q, R are collinear.

Test of Coplanarity : Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, x + y + z + w = 0.

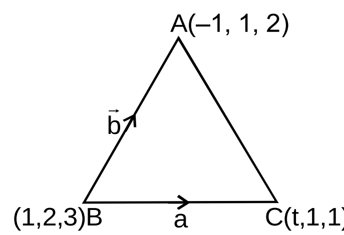
Ex.24 $\hat{u}$ and $\hat{v}$ are two non-colinear unit vectors such that $\left|\frac{\hat{u}+\hat{v}}{2} + \hat{u}\times\hat{v}\right| = 1$. Prove that $|\hat{u} \times \hat{v}| = \left|\frac{\hat{u}+\hat{v}}{2}\right|$.

Sol. Given that $\left|\frac{\hat{u}+\hat{v}}{2} + \hat{u}\times\hat{v}\right| = 1 \Rightarrow \left|\frac{\hat{u}+\hat{v}}{2} + \hat{u}\times\hat{v}\right|^2 = 1 \Rightarrow \frac{2+2\cos\theta}{4} + \sin^2\theta = 1$

$\Rightarrow \quad \cos^2\frac{\theta}{2} = \cos^2\theta \Rightarrow \theta = n\pi \pm \frac{\theta}{2} \Rightarrow \theta = \frac{2\pi}{3} \quad \Rightarrow \quad |\hat{u} \times \hat{v}| = \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \left|\frac{\hat{u}+\hat{v}}{2}\right|$.

Ex.25 Let A_m be the minimum area of the triangle whose vertices are A(–1, 1, 2); B(1, 2, 3) and C(t, 1, 1) where t is a real number. Compute the value of $\left(1338\sqrt{3}\right)(A_{min})$.

Sol. $A = \frac{1}{2}|\vec{a} \times \vec{b}|$ and $|\vec{a} \times \vec{b}|^2 = \vec{a}^2\,\vec{b}^2 - (\vec{a}^2 . \vec{b}^2)^2$



$\vec{a} = (t-1)\hat{i} - \hat{j} - 2\hat{k}$; $\quad b = 2\hat{i} + \hat{j} + \hat{k}$

$|\vec{a}|^2 = (t-1)^2 + 1 + 4$; $\quad |\vec{b}|^2 = 4 + 1 + 1 = 6$

$\vec{a} \cdot \vec{b} = 2(t-1) - 1 - 2 = 2t - 5$

$|\vec{a} \times \vec{b}|^2 = 6[t^2 - 2t + 6] - (4t^2 + 25 - 20t)$

$|\vec{a} \times \vec{b}|^2 = 2t^2 + 8t + 11$ which is minimum at t = –2

$|\vec{a}\times\vec{b}|^2_{min} = 8 - 16 + 11 = 3 \quad \Rightarrow \quad |\vec{a} \times \vec{b}|_{min} = \sqrt{3} \quad \therefore \quad \frac{|\vec{a}\times\vec{b}|_{min}}{2} = A_{min} = \frac{\sqrt{3}}{2}$

hence $\left(1338\sqrt{3}\right).\frac{\sqrt{3}}{2} = 669 \times 3 = 2007$